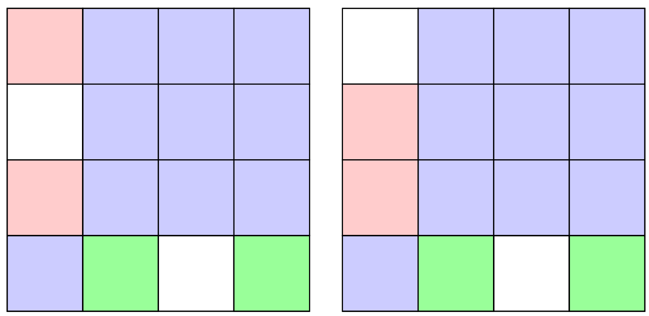
Same Colour? Don't share an Edge

We have a $n \times n$ board. We want to colour the board so that each coloured block is $m \times m$, where $m \leq n$ and blocks of the same colour cannot share an edge. However, they can share a corner. Examples are as follows:

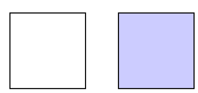


Valid Invalid

Examples of valid and invaid colouring

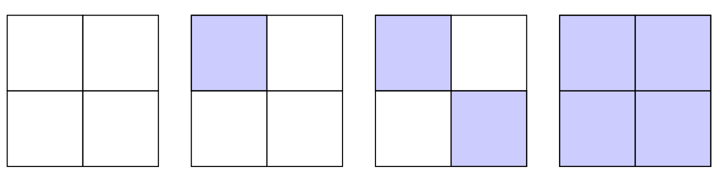
We are looking to find, for a $n \times n$ board, all possible number of squares, $F(n)$, we can fill with one singlecolour (assumed to be blue in the following.)

For $n = 1$, there is one square and it can be coloured blue or any other colour, so $F(n) = \{0, 1\}$.



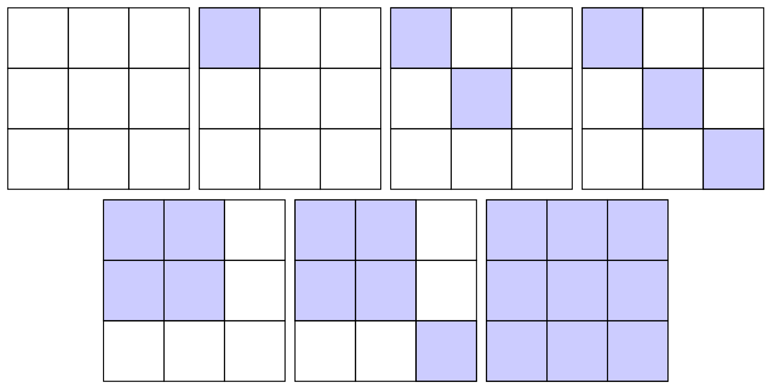
Feasible colouring for $n = 1$

For $n = 2$, the board can be filled as shown below. So $F(n) = \{0, 1, 2, 4\}$.



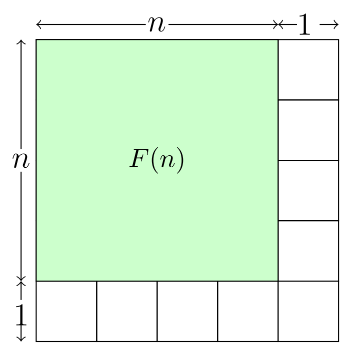
Feasible colouring for $n = 2$

For $n = 3$, the board can be filled as shown below. So $F(n) = \{0, 1, 2, 3, 4, 5, 9\}$.



Feasible colouring for $n = 3$

As illustrated below, for any $i \in F(n)$ (green square), $i \in F(n+1)$ — by colouring the additional sqaures by colours other than blue.



It would appear that $F(n) = \{0, 1, ..., (n-1)^2, (n-1)^2+1, n^2\}$. However, for $n = 7, 8$, we have the following:

1	2	3	4	5		
6	7	8	9	10		30
11	12	13	14	15		
16	17	18	19	20		31
21	22	23	24	25		
					26	27
	32		33		28	29

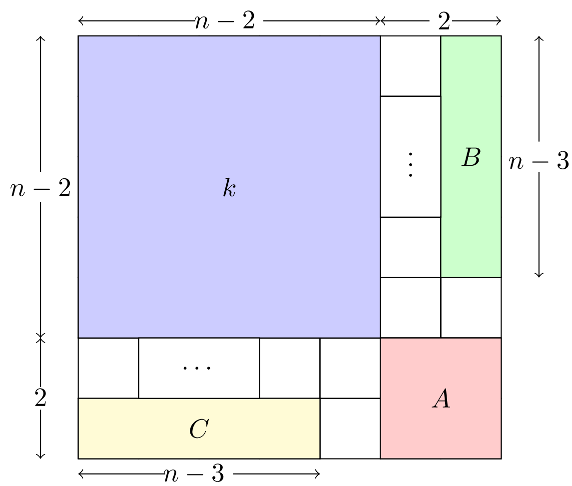
1	2	3	4	5	6	
7	8	9	10	11	12	
13	14	15	16	17	18	
19	20	21	22	23	24	
25	26	27	28	29	30	
31	32	33	34	35	36	
						37

1	2	3	4	5	6		41
7	8	9	10	11	12		
13	14	15	16	17	18		42
19	20	21	22	23	24		
25	26	27	28	29	30		43
31	32	33	34	35	36		
						37	38
44		45		46		39	40

1	2	3	4	5	6	7	
8	9	10	11	12	13	14	
15	16	17	18	19	20	21	
22	23	24	25	26	27	28	
29	30	31	32	33	34	35	
36	37	38	39	40	41	42	
43	44	45	46	47	48	49	
							50

Feasible colouring for $n = 7, 8$ with $\geq (n-2)^2$ squares coloured in blue

Notice that $34, 35 \notin F(7)$ and $47, 48 \notin F(8)$. In general, let $k = F(n-2)$ we have the following:



Colouring demonstration of $\geq (n-2)^2$ squares coloured in blue

We can consider A as a 2×2 board. So we can colour 0, 1, 2 or 4 squares in A blue. We can colour 2 squares in A and one additional square in one of B, C. Thus $(n-2)^2 + i \in F(n)$ for $0 \leq i \leq 4$.

Since B and C are $(n-3) \times 1$ and $1 \times (n-3)$ respectively, we can colour alternate squares in each of B and C blue, starting from the square closest to A. So we can colour up to $\lceil \frac{n-3}{2} \rceil$ squares blue in each. Therefore $(n-2)^2 + i \in F(n)$ for $4 \leq i \leq 2\lceil \frac{n-3}{2} \rceil$ and we have

$$i \in F(n) \text{ for } i = \begin{cases} 0, 1, ..., (n-2)^2 \\ (n-2)^2 + 1, ..., (n-2)^2 + 4 + 2\lceil \frac{n-3}{2} \rceil \\ (n-1)^2, (n-1)^2 + 1, n^2 \end{cases} \tag{1}$$

For $n \leq 6$, we have

$$\begin{aligned} (n-2)^2 + 4 + 2\left\lceil \frac{n-3}{2} \right\rceil &= n^2 - 4n + 4 + 4 + 2\left\lceil \frac{n-3}{2} \right\rceil \\ &= (n^2 - 2n + 1) - 2n + 3 + 4 + 2\left\lceil \frac{n-3}{2} \right\rceil \\ &= \begin{cases} (n-1)^2 - 2n + 3 + 4 + 2\frac{n-3}{2} & \text{if } n \text{ odd} \\ (n-1)^2 - 2n + 3 + 4 + 2\frac{n-2}{2} & \text{if } n \text{ even} \end{cases} \\ &= \begin{cases} (n-1)^2 - n + 4 & \text{if } n \text{ odd} \\ (n-1)^2 - n + 5 & \text{if } n \text{ even} \end{cases} \\ &\geq (n-1)^2 - 1 \end{aligned}$$

Thus (1) applies for $n \leq 6$.

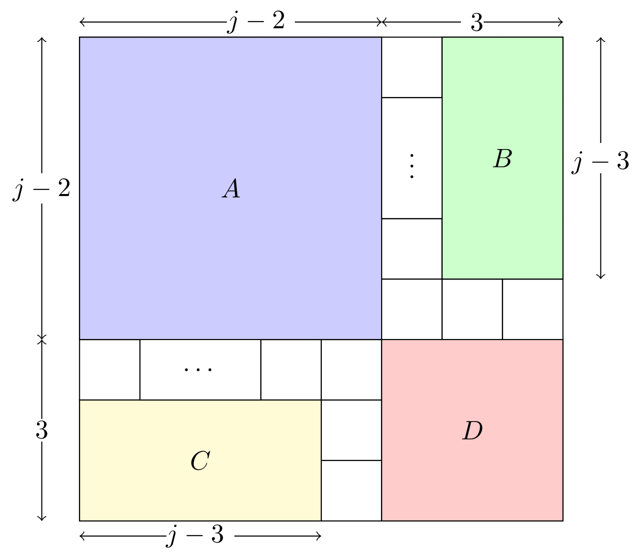
To show that (1) applies for all n, suppose (1) holds for all $n \leq j$ for some $j \geq 1$. We now derive $F(j+1)$. We have $F(j) = \{0, 1, ..., (j-2)^2, (j-2)^2 + 1, ..., (j-2)^2 + 4 + 2\lceil \frac{j-3}{2} \rceil, (j-1)^2, (j-1)^2 + 1, j^2\}$.

As shown above, we have the following:

1. $i \in F(j+1) \ \forall \ i \in F(j)$
2. $(j-1)^2 + 2, (j-1)^2 + 3, ..., (j-1)^2 + 4 + 2\lceil \frac{j+1-3}{2} \rceil \in F(j+1)$
3. $j^2 + 1, (j+1)^2 \in F(j+1)$

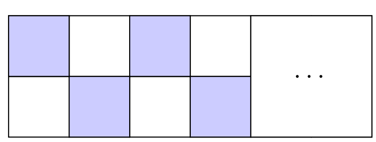
So we need to find check if $i \in F(j+1)$ for $(j-2)^2 + 5 + 2\lceil \frac{j-3}{2} \rceil \leq i \leq (j-1)^2 - 1\}$.

Consider the following:



Feasible colouring with $\geq (j-2)^2$ squares coloured in blue

We can colour alternate squares in each column of B and each row of C as follows:



Colouring demonstration squares coloured in blue for C

We can colour up to $j-3$ squares in each of B, C. Let l be the total number of squares coloured blue in A, B, C. By colouring all squares in A blue, we have

$$\begin{aligned}(j-2)^2 \leq l &\leq (j-2)^2 + 2(j-3)\\ &= j^2 - 4j + 4 + 2j - 6\\ &= j^2 - 2j + 1 - 3\\ &= (j-1)^2 - 3\end{aligned}$$

So $l \in F(j+1)$ for $(j-2)^2 \leq l \leq (j-1)^2 - 3$. Now D can be considered a 3×3 board and $a \in F(3)$ for $a = 0, 1, 2$. Therefore we have $i \in F(j+1)$ for $(j-2)^2 \leq i \leq (j-1)^2 - 3 + a \leq (j-1)^2 - 3 + 2 = (j-1)^2 - 1$. So we have

$$i \in F(j+1) \text{ for } i = \begin{cases} 0, 1, ..., (j-1)^2 \\ (j-1)^2 + 1, ..., (j-1)^2 + 4 + 2\lceil \frac{j+1-3}{2} \rceil \\ j^2, j^2 + 1, (j+1)^2 \end{cases}$$

Thus equation (1) holds for $j+1$ and therefore all $n \geq 1$.