The Last One Standing

Suppose n people are playing a game, where they line up in a circle and face the same direction (clockwise or counterclockwise). The players are assigned numbers 1 to n, with player 1 standing behind player 2, who stands behind player 3, and so forth. (So player n stands behind player 1). The game involves people tagging the person directly in front of them and the player being tagged removed from the game. The last player remaining in the game would be the winner. For the players, the natural question is "where should I stand to guarantee victory". (In the following we assume that the game starts with Player 1 tagging Player 2.)

Any even-numbered player has no chance of winning – Player 2 is tagged by Player 1, Player 4 is tagged by Player 3 and so forth.

In a game with n players, let $m = \lfloor log_2\ n \rfloor$. Then winner would be player i, where

$$i = 2n - 2^{m+1} + 1$$

Proof: We show this holds for any n by induction on m.

If $n = 1$, $m = 0$. We have the following:

$$\begin{aligned} i &= 2 \cdot 1 - 2^{\lfloor log_2\ 1 \rfloor + 1} + 1 \\ &= 2 - 2^1 + 1 \\ &= 1 \end{aligned}$$

So Player 1 would be the winner (by default).

If $n = 2$, $m = 1$. We have the following:

$$\begin{aligned} i &= 2 \cdot 2 - 2^{\lfloor log_2\ n \rfloor + 1} + 1 \\ &= 4 - 2^2 + 1 \\ &= 1 \end{aligned}$$

So Player 1 would be the winner.

If $n = 3$, $m = 1$. We have the following:

$$\begin{aligned} i &= 2 \cdot 3 - 2^{\lfloor log_2\ n \rfloor + 1} + 1 \\ &= 6 - 2^2 + 1 \\ &= 3 \end{aligned}$$

So Player 3 would be the winner.

For the induction hypothesis, suppose, for some positive integer k, the formula holds for all $n < 2^k$. Note that $k = 2^{log_2 k}$.

Now suppose we have a game with n players, where $2^k \leq n < 2^{k+1}$.

If n is even, then we have Player 1 tagging Player 2, Player 3 tagging Player 4, and so forth. After Player $n-1$ tags Player n, there are $n' = \frac{n}{2}$ players remaining. We can consider this a game with n' players and renumber the remaining players as follows:

Player Number	
Original	New
1	1
3	2
5	3
$\vdots$	$\vdots$
$n-1$	n'

Player i in the original game would be Player $i' = \frac{(i+1)}{2}$ in the new game.

Now the formula can be applied to identify the winner of the new game, as $n' = n/2 < 2^{k+1}/2 = 2^k$. The winner of the new game would be:

$$i' = 2 \cdot n' - 2^{\lfloor log_2\ n' \rfloor + 1} + 1$$

So i can be calculated as follows:

$$\begin{aligned}
i' &= 2 \cdot n' - 2^{\lfloor log_2\ n' \rfloor + 1} + 1 \\
\frac{i+1}{2} &= 2 \cdot \frac{n}{2} - 2^{\lfloor log_2\ \frac{n}{2} \rfloor + 1} + 1 \\
i + 1 &= 2 \cdot n - 2 \cdot 2^{\lfloor log_2\ \frac{n}{2} \rfloor + 1} + 2 \\
i + 1 &= 2 \cdot n - 2 \cdot 2^{\lfloor log_2\ n \rfloor - 1 + 1} + 2 \\
i &= 2 \cdot n - 2 \cdot 2^{\lfloor log_2\ n \rfloor} + 1 \\
i &= 2 \cdot n - 2^{\lfloor log_2\ n \rfloor + 1} + 1
\end{aligned}$$

If n is odd, then we have Player 1 tagging Player 2, Player 3 tagging Player 4, ... Player $n-2$ tagging Player $n-1$, and Player n tagging Player 1. After Player 1 is tagged, Players 3, 5, 7, ... n remain. So there are $n' = \frac{n-1}{2}$ players remaining. We can consider this a game with n' players and renumber the remaining players as follows:

Player Number	
Original	New
3	1
5	2
7	3
$\vdots$	$\vdots$
n	n'

Player i in the original game would be Player $i' = \frac{i-1}{2}$ in the new game.

Now the formula can be applied to identify the winner of the new game, as $n' = \frac{(n-1)}{2} < \frac{n}{2} < \frac{2^{k+1}}{2} = 2^k$. The winner of the new game would be:

$$i' = 2 \cdot n' - 2^{\lfloor log_2\ n' \rfloor + 1} + 1$$

Since n is odd, $log_2 n$ is not an integer. So $\lfloor log_2 n \rfloor = \lfloor log_2(n-1) \rfloor$. So i can be calculated as follows:

$$
\begin{aligned}
i' &= 2 \cdot n' - 2^{\lfloor log_2 \ n' \rfloor + 1} + 1 \\
\frac{i-1}{2} &= 2 \cdot \frac{n-1}{2} - 2^{\lfloor log_2 \ \frac{n-1}{2} \rfloor + 1} + 1 \\
i - 1 &= 2 \cdot (n-1) - 2 \cdot 2^{\lfloor log_2 \ \frac{n-1}{2} \rfloor + 1} + 2 \\
i - 1 &= 2 \cdot n - 2 \cdot 2^{\lfloor log_2 \ (n-1) \rfloor - 1 + 1} \\
i &= 2 \cdot n - 2 \cdot 2^{\lfloor log_2 \ n \rfloor} + 1 \\
i &= 2 \cdot n - 2^{\lfloor log_2 \ n \rfloor + 1} + 1
\end{aligned}
$$

So the formula holds for $2^k \leq n < 2^{k+1}$.

Since k can be any positive integer, the formula holds for any k and therefore for all $n > 0$.

$\square$