Making a Standing Count

At the beginning of a season, the supporters of each team wonder how their favourite team will fare in the upcoming season. While most people hope for a season in which the team emerges victorious in every game, the players, coaches, fans set the number of victories/points the team will gain. This can lead someone (me?) to wonder the number of possible ways a team's record can be at the conclusion of a season. (In the following, assume a team plays n games a season.)

If the table/standing only displays wins and losses, then there are $(n+1)$ possible combinations of wins and losses. (For any integer $i = 0, 1, \cdots, n$, if a team wins i games, it loses $n-i$ games. So we have $(n+1)$ combinations.)

For leagues where games can end in draws, we can compute the number of win-draw-loss combinations as follows: first assume that a team draws i games in the season, then there are $n-i$ games in which the team wins or loses. Then we can use the argument in the previous paragraph to deduce that there are $(n-i+1)$ possible win-loss combinations. Now i can be any integer between 0 and n. So the number of possible win-draw-loss combinations in a season in which a team plays n games is:

$$\begin{aligned}
\sum_{i=0}^{n}(n+1-i) &= \sum_{i=0}^{n}(n+1) - \sum_{i=0}^{n} i \\
&= n^2 + 2n + 1 - \frac{n^2+n}{2} \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \binom{n+2}{2}
\end{aligned}$$

In cases where a game can end in four possible ways, we can compute the number of ways a team's record can appear in the standings as follows: we first name the four possible outcomes A, B, C, and D. Then we assume that the team gets result A in j games. For the remaining $n-j$ games, the result is one of B, C, or D. Using the reasoning for three possible outcomes and the fact that j is an integer between 0 and n, we can see that the number of possible ways a team's record can appear in the final standings is:

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\begin{aligned}
\sum_{j=0}^{n}\sum_{i=0}^{j}(j+1-i) &= \sum_{j=0}^{n}(\frac{j^2}{2}+\frac{3j}{2}+1) \\
&= \frac{1}{2}\sum_{j=0}^{n} j^2 + \frac{3}{2}\sum_{j=0}^{n} j + \sum_{j=0}^{n} 1 \\
&= \frac{n(n+1)(2n+1)}{12} + \frac{3n(n+1)}{4} + (n+1) \\
&= (n+1)\left(\frac{2n^2+n}{12} + \frac{3n}{4} + 1\right) \\
&= (n+1)\left(\frac{n^2}{6} + \frac{5n}{6} + 1\right) \\
&= (n+1)\frac{(n+2)(n+3)}{6} \\
&= \binom{n+3}{3}
\end{aligned}
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