When the Sum and the Product are One and the Same

One question that appears in brain teaser books is the following: find two integers a and b so that $a \cdot b = a + b$. There are two solutions to this problem: $a = b = 0$ or $a = b = 2$.

However, if we allow a, b to be rational numbers (i.e. fractions), then there is an infinite number of solutions to this equation. From the following we can see, with one exception, that for any rational number a, there exists a rational number b such that $a \cdot b = a + b$.

We suppose that a, b are rational numbers such that $a \cdot b = a + b$. Then we have the following:

$$\begin{aligned} a \cdot b - b &= a \\ b(a-1) &= a \\ b &= \frac{a}{a-1} \end{aligned}$$

So if $a \neq 1$, there exists a unique rational number b so that $a \cdot b = a + b$.

To show that b is unique, suppose there exists a rational number c so that $b \neq c$, $a \cdot b = a + b$, and $a \cdot c = a + c$. So $b = \frac{a}{a-1}$ and $c = \frac{a}{a-1}$, a contradiction.

We now show that the only integer solutions to this equation are $a = b = 0$ and $a = b = 2$. Suppose there is another integer solution to this equation, so a is an integer other than 0 or 2.

Suppose $a = 1$, then we have $a + 1 = a$, which is impossible. Now if $a > 0$, then $a > 2$. Since $a > 2$, $a = 1$ $(mod\ a-1)$. Since $a > 2$, this means that a is not divisible by $(a-1)$. Thus $\frac{a}{a-1}$ is not an integer.

If $a < 0$, then $a - 1 < 0$. So $0 < \frac{a}{a-1}$. Since $a < 0$, $|a| < |a-1|$ and $\frac{a}{a-1} = \frac{|a|}{|a-1|}$ (as a and $(a-1)$ have the same sign). So $\frac{a}{a-1} < 1$.

In either case, a contradiction occurs. So the only integer solutions to $a \cdot b = a + b$ are $a = b = 0$ and $a = b = 2$.

Now we consider the problem of finding integers a, b, c so that $a \cdot b \cdot c = a + b + c$.

There are an infinite number of solutions to this problem–we have a solution if we set a to be any integer, $b = -a$, and $c = 0$. However, if we add the restriction that none of a, b, c can be 0, then there are only two solutions to this problem: $(a, b, c) = (3, 2, 1)$ *or* $(-3, -2, -1)$.

To show that the above triplets are the only solutions to the problem $a \cdot b \cdot c = a + b + c$. We first consider the case where a, b, c are rational numbers. For all rational numbers a, b, there exists a unique rational number c such that $a \cdot b \cdot c = a + b + c$.

Given rational numbers a, b, we can find c so that $a \cdot b \cdot c = a + b + c$, as shown below.

$$\begin{aligned}
a \cdot b \cdot c &= a + b + c \\
a \cdot b \cdot c - c &= a + b \\
c(a \cdot b - 1) &= a + b \\
c &= \frac{a + b}{a \cdot b - 1}
\end{aligned}$$

So if $a \cdot b \neq 1$, there exists rational number c such that $a \cdot b \cdot c = a + b + c$.

The proof of the uniqueness of c is as follows: given a, b, suppose there exists rational numbers c, d such that $c \neq d$, $a \cdot b \cdot c = a + b + c$ and $a \cdot b \cdot d = a + b + d$. Then $c = \frac{a+b}{a \cdot b - 1}$ and $d = \frac{a+b}{a \cdot b - 1}$. Thus $c = d$ (a, b are fixed), a contradiction.

We now prove that when none of a, b, c is 0, the only integer solutions to the equation $a \cdot b \cdot c = a + b + c$ are (a, b, c) = (3,2,1) or (-3,-2,-1).

From above, if $c = \frac{a+b}{a \cdot b - 1}$, then the triple (a, b, c) is a solution to $a \cdot b \cdot c = a + b + c$. So if $(a + b)$ is divsibile by $a \cdot b - 1$, integers a, b, c form a solution to $a \cdot b \cdot c = a + b + c$.

We first prove a couple of statements.

1. If a, b, c is a solution to the equation $a \cdot b \cdot c = a + b + c$, then so is $-a, -b, -c$.

2. If non-zero integers a, b, c form a solution to the equation $a \cdot b \cdot c = a + b + c$, then a, b, c have the same sign.

The proofs are as follows:

1.

$$\begin{aligned}
(-a) \cdot (-b) \cdot (-c) &= -(a \cdot b \cdot c) \\
&= -(a + b + c) \\
&= (-a) + (-b) + (-c) \quad \square
\end{aligned}$$

2. Suppose there exist non-zero integers a, b, c such that $a \cdot b \cdot c = a + b + c$ and not all of them have the same sign.

   If a, b are both positive, $a + b$ is also positive. Now a, b cannot both be equal to 1 (since $a \cdot b \neq 1$). So $a \cdot b > 1$. Therefore $(a \cdot b - 1)$ is positive and thus c is positive. Similarly, if a, b are both negative, then so is c.

   So exactly one of a, b is positive and the other is negative. Also, $|a| \neq |b|$ as otherwise we would have $a + b = 0$ and $c = \frac{0}{a \cdot b - 1} = 0$, which violates the non-zero requirement. Without loss of generality, assume $|a| > |b|$ (otherwise switch the signs of a and b).

   We consider the number

   $$|c| = |\frac{a + b}{a \cdot b - 1}| = \frac{|a + b|}{|a \cdot b - 1|}.$$

As a is positive and b is negative, $|a+b| = |a| - |b| < |a|$. Since $a + b \neq 0$, $|a+b| > 0$.

Since a and b have opposite signs, $a \cdot b$ is negative. As $a \cdot b < 0$, $|a \cdot b - 1| > |a \cdot b| = |a| \cdot |b| \geq |a|$ (since $|b| \geq 1$).

So $0 < |a+b| < |a| < |a \cdot b - 1|$. Thus $|c| = \frac{|a+b|}{|a \cdot b - 1|} < 1$ and $|c| > 0$. So $-1 < c < 1$ and $c \neq 0$, therefore c is not an integer, a contradiction.

Thus a, b, c have the same sign. □

So we now only need to consider a, b, c where they are all positive. If $a = b$, then $c = \frac{2 \cdot a}{a^2 - 1}$. Thus $2 \cdot a \geq a^2 - 1$. So we have $0 \geq a^2 - 2 \cdot a - 1$.

By the quadratic formula, the solutions to the equation $0 = a^2 - 2 \cdot a - 1$ are $1 \pm \sqrt{2}$. Since we have $0 \geq a^2 - 2 \cdot a - 1 = (a - 1 + \sqrt{2})(a - 1 - \sqrt{2})$, exactly one of $(a - 1 + \sqrt{2})$ and $(a - 1 - \sqrt{2})$ is non-positive. So $1 + \sqrt{2} \geq a \geq 1 - \sqrt{2}$. Since a is an integer and $\sqrt{2}$ is not, $a = 0, 1, or\ 2$.

a is positive, so $a \neq 0$. If $a = 1$, then $c = 2 \cdot a/(a^2 - 1) = 2/0$, which is impossible. This leaves $a = 2$. However, this means that $c = 4/3$, which is not an integer. Thus the equation $a \cdot b \cdot c = a + b + c$ has no solution when $a = b$.

If $a \neq b$, then we can assume, without loss of generality, that $a > b$.

First we consider the case where $b = 1$. Then we have

$$c = \frac{a+1}{a-1} = 1 + \frac{2}{a-1}.$$

Since c is an integer, $(a - 1) = 1\ or\ 2$. So $a = 2\ or\ 3$. If $a = 2, c = 3$; if $a = 3, c = 2$. (If $a > 3$, then $0 < \frac{2}{a-1} < 1$. Thus c is not an integer for $a > 3$.)

Now we consider b where $b \geq 2$. We have $a + b < 2 \cdot a$ (since $a > b$). So $a + b \leq 2 \cdot a - 1$. Since $b \geq 2$, $a \cdot b - 1 \geq 2 \cdot a - 1$. Thus $a \cdot b - 1 \geq a + b$. We also need $a + b$ to be divisible by $a \cdot b - 1$, so $a + b \geq a \cdot b - 1$ (as $a + b > 0$). So $a + b = a \cdot b - 1 = 2 \cdot a - 1$.

Since $a \cdot b - 1 = 2 \cdot a - 1$ and $a \neq 0$, $b = 2$. Since $a + b = 2 \cdot a - 1$, $b = a - 1$. So $a = 3$ and $c = 1$.

In all cases, we have a permutation of (3,2,1). So the only possible triples (a, b, c) where a, b, c are all non-zero and satisfy $a \cdot b \cdot c = a + b + c$ are (3,2,1) and (-3,-2,-1) (by statement 1).

In fact, for any integer $n \geq 2$, there is an integer solution to the equation

$$a_1 + a_2 + \cdots + a_n = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

such that $a_1, a_2, \cdots a_n$ are all non-zero — take $a_1 = n, a_2 = 2$, and $a_i = 1$ for $i = 3, 4, \cdots, n$.