What is the sum of the first n perfect squares, cubes, etc.?

According to legend, while in elementary school, a teacher of Carl Friedrich Gauss asked the students to add the integers from 1 to 100. Gauss came up with the answer within seconds, realizing that the integers can be partitioned into 50 pairs: (1, 100), (2, 99), ... (50, 51) so that the sum of the numbers in all pairs are identical. While most of the story is true, the actual problem the teacher gave was a more difficult one.

It is possible to find $S_k(n)$ the sum of the k-th powers of the first n positive integers. All it takes is a little recursion.

By the binomial theorem, we know that, for any n,

$$(n+1)^{(k+1)} = n^{(k+1)} + \binom{k+1}{1} \cdot n^k + \binom{k+1}{2} \cdot n^{(k-1)} + \cdots + 1$$

$$(n+1)^{(k+1)} - n^{(k+1)} = \binom{k+1}{1} \cdot n^k + \binom{k+1}{2} \cdot n^{(k-1)} + \cdots + 1$$

So we have the following:

$$2^{(k+1)} - 1^{(k+1)} = (k+1)1^k + \binom{k+1}{2}1^{(k-1)} + \cdots + 1$$

$$3^{(k+1)} - 2^{(k+1)} = (k+1)2^k + \binom{k+1}{2}2^{(k-1)} + \cdots + 1$$

$$4^{(k+1)} - 3^{(k+1)} = (k+1)3^k + \binom{k+1}{2}3^{(k-1)} + \cdots + 1$$

$$\vdots$$

$$(n+1)^{(k+1)} - n^{(k+1)} = (k+1)n^k + \binom{k+1}{2}n^{(k-1)} + \cdots + 1$$

Adding these equations and we get

$$(n+1)^{(k+1)} - 1 = (k+1)\sum_{i=1}^{n} i^k + \binom{k+1}{2}\sum_{i=1}^{n} i^{(k-1)} + \cdots + \sum_{i=1}^{n} 1$$

$$(n+1)^{(k+1)} - 1 = (k+1)S_k(n) + \binom{k+1}{2}S_{k-1}(n) + \cdots + S_0(n)$$

where

$$S_j(n) = \sum_{i=1}^{n} i^j$$

for $j = 0, 1, \cdots k$.

This means that we can find the sum $S_k(n)$ for any k recursively.

The polynomial that yields the sum $S_k(n)$ for k from 0 to 5 are listed below.

k	$S_k(n)$
0	n
1	$\frac{n^2+n}{2}$
2	$\frac{2n^3+3n^2+n}{6}$
3	$\frac{n^4+2n^3+n^2}{4}$
4	$\frac{6n^5+15n^4+10n^3-n}{30}$
5	$\frac{2n^6+6n^5+5n^4-n^2}{12}$

More generally, given integers a, b, one can use the above formula to compute the sum of the n-th power of integers $a, a+1, a+2, \cdots b$. (For any k, $S_k(0) = 0$.)

Case	Sum of powers between a and b inclusive
$a, b > 0$	$S_k(b) - S_k(a-1)$
$a, b < 0$	$S_k(\|a\|) - S_k(\|b\|-1)$
$a < 0, b > 0$	
- n even	$S_k(b) + S_k(\|a\|)$
- n odd	$S_k(b) - S_k(\|a\|)$

An Aside: Unfortunately, this does not have many applications. For $k = 0$, the formula gives the number of integers between 1 and n inclusive (a little redundant). For $k = 1$, the formula gives the n-th triangle number. Meanwhile, $S_2(n)$ yields the number of golf balls required to build a square pyramid that has n levels (interesting, but not very practical except maybe at parties).