Have a Pick-up Game

Suppose you are playing a game with another person. The rules are as follows:

- There are a objects on the table.
- Both players take turns picking objects from the table. In each turn, a player can take $1, 2, \cdots$ *or* b objects for some positive integer b.
- The player taking the final object wins.

The goal, naturaly, is to come up with a strategy which guarantees victory. The good news is that there is one.

Let $c = a \ (mod \ b+1)$. Assuming you pick first, during the first turn, you should pick c objects. For the rest of the game, if the other player picks d objects during a turn, you should pick $b+1-d$ objects in the next turn. (So if the other player picks 2 objects in the last turn, you should pick $b-1$ objects.)

However, if a is divisible by $b+1$ (or $c=0$), the player picking second would win by picking $b+1-d$ objects immediately after the other player picks d objects. If you pick first you would have to wait until the number of objects remaining is not divisible by $b+1$ after the other player's turn.

If the player taking the final object loses and you pick first, you should pick $c-1$ objects in the first turn. Then apply the above strategy for the rest of the game. (If $c=1$, you should pick second.)

From the above, we can see that you should pick second when $c=0$ and the player taking the final object wins; or when $c=1$ and the player taking the final object loses. Otherwise you should pick first.

Suppose the player picking the final object wins. Since $c = a \ (mod \ b+1)$, there are $a = k(b+1)+c$ objects at the beginning of the game for some positive integer k. Assume you pick first and you pick $c \neq 0$ objects during the first turn, then there would be $k(b+1)$ objects remaining. During the next turn, the other player would pick d objects and you would pick $b+1-d$ objects. So $d+(b+1-d) = b+1$ objects would be removed and there would be $(k-1)(b+1)$ objects remaining on the table (after your second turn). By following the above strategy, there would be $(k-i)(b+1)$ objects remaining on the table after your $i+1$-th turn. Thus, no objects would remain on the table after your $k+1$-th turn, handing you the victory.

If $c=0$, you are assured victory by picking second. During the i-th turn, the other player picks d objects and you pick $b+1-d$ objects, leaving $(k-i)(b+1)$ objects on the table. So you are able to remove all objects left on the table during your k-th term, thus winning the game.

Now suppose the player picking the final object loses and $c \neq 1$. If you pick $c-1$ objects in the first turn, there would be $k(b+1)+1$ objects remaining. During the next turn, the other player would pick d objects and you would pick $b+1-d$ objects. So $d+(b+1-d) = b+1$ objects would be removed and there would be $(k-1)(b+1)+1$ objects remaining on the table (after your second turn). By following the above strategy, there would be $(k-i)(b+1)+1$ objects remaining on the table after your $i+1$-th turn. Thus, 1 object would be left on the table after your $k+1$-th turn. The other player would have to pick up the object, giving you the victory.

If $c=1$, you are assured victory by picking second. During the i-th turn, the other player picks d objects and you pick $b+1-d$ objects, leaving $(k-i)(b+1)+1$ objects on the table. By following this strategy, there would be 1 object left on the table after your k-th term, securing the win.