In Search of (Multiplicative) Perfection

A multiplicative perfect number n is a positive integer where the product of its (positive) integer divisors is n^2.

All multiplicative perfect numbers can be expressed as one of the two following forms: p^3 or $p \cdot q$, where p, q are distinct prime numbers.

Before showing that all multiplicative perfect numbers are in either one of these two forms, we first prove the following statement:

Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_m^{a_m}$ be the prime factorization of n (i.e. $p_i \neq p_j$ for $i \neq j$).

Then the product of the divisors of n is

$$\prod_{i=1}^{n} p_i^{\frac{a_i}{2} \prod_{j=1}^{n}(a_j+1)}$$

Proof: The statement can be proven by induction on m.

For $m = 1$, let $p = p_1$ and $a = a_1$. The divisors of n are 1, $p, p^2, p^3, \cdots p^a$.

So the product of the divisors is

$$\begin{aligned} 1 \cdot p \cdot p^2 \cdot \cdots \cdot p^a &= p^{\sum_{k=0}^{a} k} \\ &= p^{\frac{a}{2} \cdot (a+1)} \end{aligned}$$

For the induction hypothesis, suppose that the statement holds for any positive integer with r distinct prime factors. We then proceed to show that the statement also holds for positive integers with $r + 1$ distinct prime factors.

Let n be a positive integer with r distinct prime factors. Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_r^{a_r}$ So we have $a_i \geq 1$ and $p_i \neq p_j$ for $i \neq j$.

Let N be the product of the divisors of n. By the induction hypothesis, we have

$$N = \prod_{i=1}^{n} p_i^{\frac{a_i}{2} \prod_{j=1}^{n}(a_j+1)}$$

Let $k = n \cdot p_{(r+1)}^{a_{(r+1)}}$. So k has $r + 1$ prime divisors.

Now the divisors of k are in the form

$$p_1^{b_1} \cdot p_2^{b_2} \cdot p_r^{a_r} \cdot p_{r+1}^{a_{r+1}}$$

where $0 \leq b_i \leq a_i$ for $i = 1, 2, \cdots r + 1$.

Now for each $b_{(r+1)}, i = 0, 1, \cdots, a_{(r+1)}$, the divisor is in the form

$$\prod_{i=1}^{r+1} p_i^{b_i} = (p_{r+1})^{b_{r+1}} \cdot \prod_{i=1}^{r} p_i^{b_i}$$

So the product of these divisors is

$$(p_{r+1})^{b_{r+1} \cdot \prod_{j=1}^{r}(a_j+1)} \cdot N$$

by the induction hypothesis and that n has $(a_1+1)\cdot(a_2+2)\cdots(a_r+1)$ divisors. (This is derived from that, for $i = 1, 2, \cdots, r$, there are $(a_i + 1)$ choices for b_i — $0, 1, 2, \cdots, a_i$.)

So the product of the divisors of k, P, is

$$\begin{aligned}
P &= \prod_{b_{r+1}=0}^{a_{r+1}} (p_{r+1})^{b_{r+1} \cdot \prod_{j=1}^{r}(a_j+1)} \cdot N \\
&= (p_{r+1})^{\sum_{b_{r+1}=0}^{a_{r+1}} b_{r+1} \cdot \prod_{j=1}^{r}(a_j+1)} \cdot N^{(a_{r+1}+1)} \\
&= (p_{r+1})^{\frac{a_{r+1}\cdot(a_{r+1}+1)}{2} \prod_{j=1}^{r}(a_j+1)} \cdot \left(\prod_{i=1}^{r} p_i^{\frac{a_i}{2} \cdot \prod_{j=1}^{r}(a_j+1)} \right)^{(a_{r+1}+1)} \\
&= (p_{r+1})^{\frac{a_{r+1}}{2} \prod_{j=1}^{r+1}(a_j+1)} \cdot \prod_{i=1}^{r} p_i^{\frac{a_i}{2} \cdot \prod_{j=1}^{r+1}(a_j+1)} \\
&= \prod_{i=1}^{r+1} p_i^{\frac{a_i}{2} \cdot \prod_{j=1}^{r+1}(a_j+1)}
\end{aligned}$$

Thus the staement is true for any positive integer with $r + 1$ prime factors. Since r can be any positive integer, the statement holds for all positive integers. $\square$

We now return to the problem of identifying the multiplicative perfect positive integers.

If a positive integer n is multiplicative perfect, the product of the divisors of n is n^2. If $p_1^{a_1} \cdot p_2^{a_2} \cdot p_m^{a_m}$ is the prime factorization of n, then n is a multiplicative perfect number if it satisfies the following:

$$\prod_{i=1}^{n} p_i^{\frac{a_i}{2} \cdot \prod_{j=1}^{n}(a_j+1)} = \prod_{i=1}^{n} p_i^{2a_i}$$

The following has to hold for each i for n to be mulitplicative perfect.

$$\frac{a_i}{2} \cdot \prod_{j=1}^{n}(a_j + 1) = 2 \cdot a_i$$

$$\prod_{j=1}^{n}(a_j + 1) = 4$$

Since a_j are all non-negative integers and $4 = 1 \cdot 4 = 2 \cdot 2$, there are two possible solutions to the above equation:

1. $a_1 + 1 = 4$
2. $a_1 + 1 = a_2 + 1 = 2$

If we let $p = p_1, q = p_2$, then the first case corresponds to $n = p^3$ and the second case corresponds to $n = p \cdot q$, where p, q are distinct primes. Since these are the only positive integers satisfying the above equation, all multiplicative perfect (positive) integers are in the form p^3 or $p \cdot q$ for some distinct primes p, q.

Note: After this section was completed, I found an article identifying all multiplicative perfect numbers using a similar arguement as the one stated above.