Something Magical about Squares

Well, not really. The following are just a few ways to construct a magic square.

So what is a magic square? A magic square of order n is a grid of n rows and n columns where the sum of the entries in any row, column, or diagonal is the same (usually there is an additional restriction where we are only allowed to use the numbers $1, \cdots n^2$ and no two entries of the square can have the same number).

Magic Squares of Odd Order

We start with an n by n grid (n is odd). Then fill the entries of the middle row (or the $(n+1)/2$-th row) of the grid with the following:

1	$n+2$	$2n+3$	$3n+4$	$\cdots$	n^2

Table 1:

(i.e. the $j+1$-th entry= the j-th entry + $(n+1)$)

Then fill out the entries as follows: /newline If the entry in the i-th row, j-th column is x, then the entry in the $(i+1)$-th row, $(j+1)$-th column is $x+2n$. If $x+2n > n^2$, then subtract $x+2n$ by n^2. If the i-th row is the final row, consider the first row of the grid as the $(i+1)$-th row, likewise for columns.

For example, for $n = 5$, we have:

18	24	5	6	12
22	3	9	15	16
1	7	13	19	25
10	11	17	23	4
14	20	21	2	8

Table 2: A 5-by-5 magic square

This method generates a magic square of order n for any odd number n. (In fact, we can replace the 2 in the expression $x+2n$ by any number $m, m \neq 1$, that is co-prime with n to obtain a different magic square.)

Magic Squares whose order is even, but not divisible by 4

For n that is even but not divisible by 4, we can use a different method to construct a magic square of order n.

There is no magic square of order 2.

First we express n as $4m+2$, where m is a positive integer. (Since n is even and not divisible by 4, n divided by 4 gives a remainder of 2.)

We start with an empty $n-by-n$ grid and we use the symbol $S_{i,j}$ to denote the entry on the i-th row, j-th column. ($S_{1,1}$ is the entry at the top-left corner of the grid.)

We first fill the entries the diagonals. We have $S_{i,i} = i \cdot n - (i-1)$ and $S_{i,(n-i+1)} = (i-1) \cdot n + i$.

For example, for $n = 10$, we have the following:

10									1
	19							12	
		28					23		
			37			34			
				46	45				
				56	55				
			67			64			
		78					73		
	89							82	
100									91

Table 3:

We then fill the middle 2 columns of the grid as follows:

- The entries on the $(2m+1)$-th and $(2m+2)$-th rows are already filled.
- Let $S_{1,2m+1} = 2m+1$ and $S_{1,2m+2} = (n-1) \cdot n + (2m+2)$.
- Let $S_{n,2m+1} = (n-1) \cdot n + (2m+1)$ and $S_{n,2m+2} = 2m+2$.
- For $i = 2, 3, \cdots, 2m$, we have $S_{i,2m+1} = (n-i+1) \cdot n + (2m+1)$ and $S_{i,2m+2} = S_{i,2m+1} + 1$.
- For $i = 2m+3, 2m+4, \cdots, n-1$ (or $4m+1$), $S_{i,j} = n^2 + 1 - S_{(n+1-i),j}$

So, for $n = 10 (= 2 \cdot 4 + 2$, so $m = 2)$, we have the following:

10				5	96				1
	19			85	86			12	
		28		75	76		23		
			37	65	66	34			
				46	45				
				56	55				
			67	36	35	64			
		78		26	25		73		
	89			16	15			82	
100				95	6				91

Table 4:

Now we fill the rest of entries in the first and final columns as follows:

- $S_{2m+1,1} = n^2/2 + 1$

- $S_{2m+1,n} = n^2/2 + n$
- $S_{2m+2,1} = n^2/2$
- $S_{2m+2,n} = n^2/2 - n + 1$
- For $i = 2, 4, \cdots, 2m$, and $i = 2m+3, 2m+5, \cdots n-1$, $S_{i,1} = (i-1) \cdot n + 1$ and $S_{i,n} = n^2 + 1 - S_{i,1}$
- For $i = 3, 5, \cdots 2m-1$, and $i = 2m+4, 2m+6, \cdots, n-2$, $S_{i,n} = (i-1) \cdot n + 1$ and $S_{i,1} = n^2 + 1 - S_{i,n}$

So, for $n = 10$, we have the following:

10				5	96				1
11	19			85	86			12	90
80		28		75	76		23		21
31			37	65	66	34			70
51				46	45				60
50				56	55				41
61			67	36	35	64			40
30		78		26	25		73		71
81	89			16	15			82	20
100				95	6				91

Table 5:

Next we deal with the middle 2 rows. We fill the blank entries as follows:

- For $j = 2, 3, \cdots, 2m$, $S_{i,j} = (i-1) \cdot n + j, i = 2m+1, 2m+2$
- For $j = 2m+3, 2m+4, \cdots n-1$, $S_{i,j} = n^2 + 1 - S_{i,(n+1-i)}, i = 2m+1, 2m+2$.

So for $n = 10$, the middle 2 rows look like this:

51	42	43	44	46	45	57	58	59	60
50	52	53	54	56	55	47	48	49	41

Table 6:

We fill the rest of the entries using a method called "Large-Small-Large Rule" (I have yet to come up with a better name). The method goes like this:

1. For each row $i, i = 1, 2, \cdots, 2m$, we find the smallest j where the $S_{i,j}$ is not yet assigned a value. Then we set $S_{i,j} = (n-i) \cdot n + j$. (Large)
2. Find the next empty entry on row i and set $S_{i,j} = (i-1) \cdot n + j$, where j is the column number. (Small)
3. Find the next empty entry on row i and set $S_{i,j} = (n-i) \cdot n + j$, where j is the column number. (Large)

4. We continue this pattern until the $2m$-th column is filled. (The $(2m+1)$-th and $(2m+2)$-th columns on row i are already filled.)

5. Let $S_{i,2m+3} = S_{i,2m} + 3$

6. We continue to alternate between small and large values, outlined in steps 1 and 2, until we fill out the row.

7. For $i = 2m+3, 2m+4, \cdots n$, if $S_{i,j}$ is empty, then $S_{i,j} = n^2 + 1 - S_{n+1-i,j}$

For example, in the second row of the 10-by-10 grid, the first empty entry is on the 3rd column, so $S_{2,3} = (10-2) \cdot 10 + 3 = 83$.

The next empty entry is on the 4th column, so $S_{2,4} = (2-1) \cdot 10 + 4 = 14. S_{2,7} = S_{2,4} + 3 = 14 + 3 = 17$ and $S_{2,8} = 88$.

We don't need to go any further as the row is now filled. The row would look as follows:

11	19	83	14	85	86	17	88	12	90

Table 7:

After the rest of the entries are filled, we have the following:

10	92	3	94	5	96	97	8	99	1
11	19	83	14	85	86	17	88	12	90
80	72	28	24	75	76	27	23	79	21
31	62	33	37	65	66	34	38	69	70
51	42	43	44	46	45	57	58	59	60
50	52	53	54	56	55	47	48	49	41
61	39	68	67	36	35	64	63	32	40
30	29	78	77	26	25	74	73	22	71
81	89	18	87	16	15	84	13	82	20
100	9	98	7	95	6	4	93	2	91

Table 8: A 10-by-10 magic square

We can construct a different magic square by switching the words "rows" and "columns" in the above description.

Magic Squares whose order is divisible by 4

If n is divisible by 4, then we can construct a magic square of order n as follows:

1. We express n in the form: $n = 2^a \cdot b$, where $a > 1$ (since n is a multiple of 4) and b is odd (b may be equal to 1).

2. Construct a magic square of order b (using any method).
   If a is even, then replace each entry of the magic square with the following:

$1+i$	$15+i$	$8+i$	$10+i$
$12+i$	$6+i$	$13+i$	$3+i$
$14+i$	$4+i$	$11+i$	$5+i$
$7+i$	$9+i$	$2+i$	$16+i$

where $i =$ (the current entry of the magic square - 1) $\cdot 16$.
This yields a magic square of order $4b$.
If a is odd, then we replace each entry of the magic square with the following:

$1+i$	$27+i$	$13+i$	$23+i$	$48+i$	$54+i$	$36+i$	$58+i$
$20+i$	$10+i$	$32+i$	$6+i$	$61+i$	$39+i$	$49+i$	$43+i$
$34+i$	$60+i$	$46+i$	$56+i$	$15+i$	$21+i$	$3+i$	$25+i$
$51+i$	$41+i$	$63+i$	$37+i$	$30+i$	$8+i$	$18+i$	$12+i$
$62+i$	$40+i$	$50+i$	$44+i$	$19+i$	$9+i$	$31+i$	$5+i$
$47+i$	$53+i$	$35+i$	$57+i$	$2+i$	$28+i$	$14+i$	$24+i$
$29+i$	$7+i$	$17+i$	$11+i$	$53+i$	$42+i$	$64+i$	$38+i$
$16+i$	$22+i$	$4+i$	$26+i$	$33+i$	$59+i$	$45+i$	$55+i$

Table 9:

where $i =$ (the current entry of the magic square - 1) $\cdot 64$.
This yields a magic square of order $8b$.

3. We now continue the process (by expanding a magic square of order m into a magic square of order $4m$) until we get a magic square of order n.

For example, we can use the following magic square of order 3 to construct a magic square of order 12.

8	3	4
1	5	9
6	7	2

Table 10:

The resulting magic square of order 12 is:

113	127	120	122	33	47	40	42	49	63	56	58
124	118	125	115	44	38	45	35	60	54	61	51
126	116	123	117	46	36	43	37	62	52	59	53
119	121	114	128	39	41	34	48	55	57	50	64
1	15	8	10	65	79	72	74	129	143	136	138
12	6	13	3	76	70	77	67	140	134	141	131
14	4	11	5	78	68	75	69	142	132	139	133
7	9	2	16	71	73	66	80	135	137	130	144
81	95	88	90	97	111	104	106	17	31	24	26
92	86	93	83	108	102	109	99	28	22	29	19
94	84	91	85	110	100	107	101	30	20	27	21
87	89	82	96	103	105	98	112	23	25	18	32

Table 11: A 12-by-12 magic square

In general, one can use any magic square of two arbitrary orders, say a, b, to construct a magic square of order $a \cdot b$ using this method.

References and Acknowlegements

- Eric W. Weisstein. "Magic Square." From MathWorld — A Wolfram Web Resource. http://mathworld.wolfram.com/MagicSquare.html.

- The method on constructing magic squares of odd order shown on this page was discovered by J. H. Conway, even though I arrived at this method independently.