Being Drawn Apart

In single-elimination tournaments, teams may receive byes and enter the tournament in later rounds (instead of the first round). If the draw determining the matchups for each round is held individually (ie the teams/competitors do not know who they will face in the next round until the draw for that round is made) and the round consists of teams advancing from a previous round and those entering the tournament at this round (the 3rd round of the English FA Cup, for example), one may ask how likely each team advancing from a previous round is drawn to face a team entering the tournament at the current round. In other words, the round does not feature any matchup between teams advancing from a previous round.

Suppose there are m teams advancing from the previous round and n matches in the current round, the chance that each team advancing from a previous round is drawn to face a team entering the tournament at the current round is

$$2^m \frac{\binom{n}{m}}{\binom{2n}{m}}$$

(If $m > n$, the probability is 1 — it is certain that at least one match will feature two teams advancing from a previous round.)

One may also use this formula to calculate the chance teams from the same country avoid each other in the draw of a specific round of a tournament.

Suppose the teams advancing from the previous round are denoted Teams $1, 2 \ldots m$ and the teams are drawn in this order.

Since there are n matches in this round, there are $2n$ places in the draw.

Team 1 may be placed anywhere in the draw. Since there are $2n$ places available, Team 1 has a $\frac{2n}{2n}$ chance of avoiding another team advancing from the previous round. Now Team 2 cannot be drawn to face Team 1 and there are $2n-1$ places remaining in the draw, so there are $2n-1-1 = 2n-2 = 2(n-1)$ spots available in the draw and Team 2 has a $\frac{2(n-1)}{2n-1}$ chance of avoiding Team 1. Likewise, Team 3 cannot be drawn to face either Team 1 or 2. There are $2n-2$ places remaining, so there are $2n22 = 2(n2)$ spots available in the draw and Team 3 has a $\frac{2(n-2)}{2n-2}$ chance of avoiding Teams 1 and 2.

Liekwise, team i cannot be drawn to face any of Team $1, 2 \ldots i1$. There are $2n(i1)$ places available, there are $2n(i-1)(i-1) = 2n-2(i-1) = 2\ (n-(i-1))$ spots available in the draw and Team i has a $\frac{2(n-(i-1)}{2n-(i-1)}$ chance of avoiding another team advancing from the previous round. So the chance of Team m avoiding Teams $1, 2 \ldots m-1$, is $\frac{2(n-(m-1)}{2n-(m-1)}$

So the chance that teams advancing from a previous round avoid each other in the current round is

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\begin{aligned}
\prod_{i=1}^{m} \frac{2(n-(i-1))}{2n-(i-1)} &= 2^m \prod_{i=1}^{m} \frac{n-(i-1)}{2n-(i-1)} \\
&= 2^m \frac{n!}{(n-m)!} \cdot \frac{(2n-m)!}{2n!} \\
&= 2^m \frac{n!}{m!\,(n-m)!} \cdot \frac{m!\,(2n-m)!}{2n!} \\
&= 2^m \frac{\binom{n}{m}}{\binom{2n}{m}}
\end{aligned}
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