Everything's on the Balance (Two Times Over)

There are two defective coins among the n minted. The goal is to find the defective coins using a balance as few times as possible.

To find the two defective coins among n coins, let $k = n$ and we do the following:

Step 1: Divide the coins into three subsets, named A,B,C, such that there are $k = \lfloor n/3 \rfloor$ coins in each subset. If there is any coins remaining (there will be no more than 2), set aside them for now.

Step 2: Weigh each subset of coins against the other two (A vs B, A vs C, B vs C) to determine the weight of the three subsets relative to each other. We can determine which set(s) contains the defective coins based on the results as follows:

1. If all three subsets have the same weight (the balance is level for all three weightings), then the defective coin(s) can be found from the coins set aside earlier.

2. If no two subsets have the same weight, then the heaviest subset of coins contain no defective coin while the other two contain one defective coin each.

3. If two subsets have the same weight while the third subset is heavier, the lighter two subsets contain one defective coin each.

4. If two subsets have the same weight while the third subset is lighter, the lighter subset contains at least one defective coin.

Step 3: If the weight of all three sets are equal (Case 1), we move on to the coins set aside earlier. If 1 coin is set aside, it is defective. If 2 coins are set aside, place one coin on either side of the balance and weigh them — if the balance is level, both coins are defective; if the balance is not level, the lighter coin is defective. Add the heavier coin to the special set named set D.

For cases 2 and 3, there are two subsets containing one defective coin each. We can identify the defective coin from each subset using the method used to identify one defective coin.

For case 4, remove the two heavier subsets of coins and add the extra coins set aside earlier to the special set D. Divide the lightest set of coins into three equally sized subsets and repeat Steps 1-3.

Step 4: (For Case 4 only) Once we cannot divide a set of coins into three subsets (ie there are one or two coins in the set), we weigh the coins of the set. If there is 1 coin in the set, it is defective; if there are 2 coins in the set, we weigh the coins against each other and the lighter coin is defective. Add the heavier coin (if the set has 2 coins) to the special set D. We move on to the discarded set D. Repeat Steps 1-4 on set D until both defective coins are found.

The maximum number of balance weightings needed, m, using this method is:

$$m = 3\lceil log_3\ n\rceil + ceil * log_3\ log_3\ n + 2$$

Proof: If we encounter Case 4 during an iteration, we need to divide the lightest set into 3 equally-sized subsets. During the i-th iteration, there would be $\lfloor n/3^i \rfloor$ coins in each subset. For each iteration, each set is weighted against each other (A vs B, A vs C, B vs C), so 3 weightings are needed.

If we encounter any of Cases 1-3 during the i-th iteration, we consider the following (note that the set of coins contains at least one defective coin before being subdivided):

We encounter Case 4 in the first $i-1$ iterations, so $3(i-1)$ weightings is made. During the i-th iteration, the 3 subsets are weighted against each other and thus 3 weightings are made. Thus $3i$ weightings are made thus far. Now there are at least 1 coin in each subset, so $\lfloor n/3^i \rfloor \geq 1$. As $a \geq \lfloor a \rfloor$ for any positive number, $n/3^i \geq 1$. Thus $n \geq 3^i$ and $log_3\ n \geq log_3\ (3^i) = i$. i is an integer, so $i \leq \lceil log_3\ n\rceil$.

For Cases 1-3. the number of additional weightings are outlined below:

- **Case 1:**
  Since all three sets have equal weight, all coins within each of the 3 sets are genuine. As there is at least one defective coin within the bigger set (consisting of the three sets and the coins set aside), a defective coin can be identified from the coins set aside. If 1 coin is set aside, the coin is defective, so no extra weightings is needed; if 2 coins are set aside, the coins weighted against each other, so 1 weighting is made. The lighter coin is defective and the heavier coin is added to set D, which contains 1 defective coin.
  $3i$ weightings are made before the three sets are found to have equal weight. 0 or 1 weightings are made to identify a defective coins among those set aside during this iteration. So at most $3i + 1 \leq 3\lceil log_3\ n\rceil + 1$ weightings are made before we move onto set D.

- **Cases 2 and 3:**
  Two sets, each containing 1 defective coin are identified. So we can identify the defective coin within each set using the method for identifying one defective coin.
  Now there are $\lfloor n/3^i \rfloor$ within coins each subset. Using the above method, $\lceil log_3\ (n/3^i)\rceil$ weightings are required to identify the defective coin within each of the two sets.
  $3i$ weightings are made before two sets with smaller weights are identified. For each of the two sets containing 1 defective coin, from the case of identifying one defectove coin, $\lceil log_3\ \lfloor n/3^i \rfloor\rceil$. So the number of additional weightings needed is:

$$
\begin{aligned}
2\left\lceil log_3\ floor(n/3^i)\right\rceil &\leq 2\left\lceil log_3\ (n/3^i)\right\rceil \\
&\leq 2\left\lceil log_3\ n - log_3(3^i)\right\rceil \\
&\leq 2\left\lceil log_3\ n - i\right\rceil \\
&\leq 2\left\lceil log_3\ n\right\rceil - 2i
\end{aligned}
$$

Since the two sets contain a defective coin each, all coins in set D are genuine and there is no need to weigh the coins in this set. Thus the number of weightings required is:

$$
\begin{aligned}
3i + 2\left\lceil log_3\ n\right\rceil - 2i &= 2\left\lceil log_3\ n\right\rceil + i \\
&\leq 2\left\lceil log_3\ n\right\rceil + log_3\ n + 1 \\
&\leq 3\left\lceil log_3\ n\right\rceil + 1
\end{aligned}
$$

We now consider the case where we encounter Case 4 in each iteration (Step 4). Let $p =$ the number of iterations needed before moving onto Set D, so we cannot subdivide each subset any further. Thus each subset contains at least one coin, so $n/3^p \geq 1$ and $1 \geq n/3^{p+1}$. So $3^{p+1} \geq n \geq 3^p$ and thus $p + 1 \geq log_3\ n \geq p$. Now p is an integer, so $p = \lfloor log_3\ n\rfloor < \lceil log_3\ n\rceil$.

At each iteration, 3 weightings is made. At the final iteration, an extra weighting may be made (if the lightest subset contains 2 coins). So if we encounter Case 4 in each iteration, $3p + 1 < 3\left\lceil log_3\ n\right\rceil + 1$ weightings are made.

Now we count the number of weightings needed to identify the defective coins from set D (for cases 1 and 4). (For cases 2 and 3, all coins in set D are genuine, so no weighting is required.) At most 2 coins are set aside in each iteration before the last. In the final iteration, at most 1 coin is be added to set D. So set D contains at most $2p + 1$ coins. Now p is a positive integer, so $2p + 1 \leq 3p$. As there is exactly 1 defective coin within this set, the number of weightings required to identify the defective coin is (from the case of identifying one defective coin):

$$
\begin{aligned}
\lceil log_3\ (3p)\rceil &\leq \lceil log_3\ 3 + log_3\ p\rceil \\
&\leq \lceil 1 + log_3\ (log_3\ n)\rceil \\
&= 1 + \lceil log_3\ log_3\ n\rceil
\end{aligned}
$$

Thus the maximum number of weightings needed, m is:

$$
\begin{aligned}
m &= [3\left\lceil log_3\ n\right\rceil + 1] + [1 + \lceil log_3\ log_3\ n\rceil] \\
&= 3\left\lceil log_3\ n\right\rceil + \lceil log_3\ log_3\ n\rceil + 2
\end{aligned}
$$

$\square$