Everything's on the Balance

When I was (a lot) younger, a problem was presented to me (and my classmates) during math class. The problem is as follows: n coins is minted. One of them is found to be defective (fraudulent in some versions) and this particular coin weighs less compared to all others. The goal is to find the defective coin using a balance as few times as possible.

To find the one defective coin among n coins, we do the following:

Let $k = n$ and divide the coins into three sets, called sets 1, 2, and 3 such that set 1 has the most number of coins and set 3 has the least number of coins and the difference in the number of coins between any two sets is one or zero. In other words, if $j = \lfloor k/3 \rfloor$, then the number of coins in sets 1, 2, and 3 are as follows:

	Number of Coins		
$n(mod\ 3)$	Set 1	Set 2	Set 3
0	j	j	j
1	$j+1$	j	j
2	$j+1$	$j+1$	j

Select two equally-sized sets and place each on one side of the balance. If the balance is level, then the two sets are equal in weight and the third set of coins contains defective coin. The two sets of coins on the balance are discarded. If the balance tilts towards one side, then the defective coin is contained in the set of coins placed on the other side of the balance. The set containing the defective coin is kept while the other two sets are discarded.

Let k = the number of coins in the set containing the defective coin and repeat the process until the defective coin is found.

The maximum number of balance weightings needed, m, using this method is:

$$m = \lceil log_3\ n \rceil$$

Proof: We use induction on m to show that the maximum number of weightings is the above figure.

Since $m = \lceil log_3\ n \rceil$, $m = 0$ when $n = 1$. (When 1 of 1 coin is known to be defective, there is no need to use the balance.) So equation holds for $m = 1$.

$log_3\ 3 = 1$. So $m = 1$ for $n = 2, 3$. If $n = 2$, the defective coin can be identified simply by placing one coin on each side of the balance and see which side is lighter. If $n = 3$, we place one coin on each side of the balance. If the balance is tilted towards one side, the defective coin is on the other side; if the balance is level, the defective coin is the one not on the scale. So for $n = 2$ *or* 3, at most 1 weighting is needed.

For the induction hypothesis, suppose that, for $m \leq r$, where r is a positive integer, the maximum number of weightings needed to identify the defective coin is m. Now $m = \lceil log_3\ n \rceil \leq r$. Since $m \leq \lceil log_3\ n \rceil \leq r$, the hypothesis holds for $n \leq 3^r$. Meanwhile, $\lceil log_3\ 3^{r-1} \rceil = r - 1$. So for $3^{r-1} < n \leq 3^r$, the maximum number of weightings required is r. We now show that, for $3^r < n \leq 3^{r+1}$, the maximum number of weightings needed is $r + 1$.

We begin by dividing the $n, 3^r < n \leq 3^{r+1}$ coins into three sets using the method described above. So each set would contain either $\lfloor n/3 \rfloor$ or $\lfloor n/3 \rfloor + 1$ coins. After weighting the coins once, the set containing the defective coin is kept and the other two sets are discarded. So after the first weighting, there are at most $\lfloor n/3 \rfloor + 1$ coins remaining.

Since $3^r < n$, the number of coins remaining after the first weighting is at least $\lfloor 3^r/3 \rfloor = \lfloor 3^{r-1} \rfloor = 3^{r-1}$. If there are 3^{r-1} coins remaining, then the defective coin can be identified, by the induction hypothesis, with at most $r - 1$ more weightings. Thus, in this case, the maximum number of weightings needed is r.

For $n = 3^{r+1}$, the coins are divided into 3 sets of 3^r coins each. Since one set is kept after the first weighting, there are 3^r coins remaining after the first weighting and, by the induction hypothesis, the defective coin can by identified using at most r more weightings. So, for $n = 3^{r+1}$, the maximum number of weightings needed to identify the defective coin is $r + 1$.

For $n < 3^{r+1}$, $\lfloor n/3 \rfloor + 1 < \lfloor 3^{r+1}/3 \rfloor + 1 = 3^r + 1$. So, after the first weighting, there are at most 3^r coins remaining and, by the indcution hypothesis, the defective coin can be identified with at most r more weightings. Thus the maximum number of weightings required is $r + 1$.

Thus for $3^r < n \leq 3^{r+1}$, the maximum number of weightings needed is $r + 1$. Since r is a positive integer, the maximum number of weightings needed to identify a defective coin from n coins is $m = \lceil log_3\ n \rceil$.

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