Lucky 7 and Unlucky 13

Most of us have heard of tricks used to see if an integer is divisible by a small integer (for example, if the last digit of a number is even, it is divisble by 2). The tricks for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, and 11 are quite well known. However, it seems that when the divisibility tricks are taught, the trick for divisibility by 7 is not mentioned. One explanation given is that no such trick exists. However, the contrary is true, such a trick does exist.

First, we will see the trick used to check if an integer is divisible by 7. Note that it only works for integers with 4 digits or more. For numbers smaller than 1000, we can check whether it is divisible by 7 the old-fashioned way – actually dividing the number by 7 and see if the remainder is 0.

Let's say we want to see if a is divisible by 7, where a is an integer having 4 or more digits. If a is negative, remove the negative sign so that a becomes positive. Then we express a in the following form:

$$a = b_0 + b_1 10^3 + b_2 10^6 + \cdots + b_n 10^3 n$$

where b_i are integers between 0 and 999 and $n = \left\lfloor \frac{log_{10}(a)}{3} \right\rfloor$. (In other words, if we start from the left, b_0 consists of the final 3 digits of a, b_1 the 3 digits before that, and so on.)

Let $c = \sum_{i=0}^{n} (-1)^i b_i$. If c is a multiple of 7, then a is divisible by 7. If c is still too large, repeat the process. (We should be able to reduce any number we encounter into a 3-digit number by repeating this process at most twice, as $|c| \leq 999(n+1)$. For example, we can reduce a 300-digit integer into an integer with 6 digits or fewer after just one iteration. We will see why the inequality holds later.)

To see if a number is divisible by 13, check if c is divisible by 13 instead of 7.

For example, suppose we want to find out if
$a = 444, 031, 154, 930, 525, 092, 999, 304, 411, 100, 630, 156$ is divisible by either 7 or 13.

a has 36 digits, so $n = \left\lfloor \frac{log_{10}(a)}{3} \right\rfloor = \left\lfloor \frac{35}{3} \right\rfloor = 11$ and $b_0 = 156$, $b_1 = 630$, $b_2 = 100$, $b_3 = 411$, $b_4 = 304$, $b_5 = 999$, $b_6 = 092$, $b_7 = 525$, $b_8 = 930$, $b_9 = 154$, $b_10 = 031$, $b_11 = 444$.

So $c = 156-630+100-411+304-999+92-525+930-154+31-444 = -1550$.

We can either repeat the process or divide c by 7 and 13 to see if it is a multiple of 7 and/or 13. In this case, we divide c directly. If we divide the $c =$ -1550 by 7 and 13, the respective remainders are both -3. So a is not divisible by either 7 or 13.

We now look at how this trick works.

Proof: Consider the number 1001. Its prime factorization is $7 \cdot 11 \cdot 13$. Since 1001 is a multiple of 7, $1000 = 6 \ (mod\ 7) = -1 \ (mod\ 7)$. So $10^{3m} = 1000^m = (-1)^m \ (mod\ 7)$. Now $10^{3m} = 1 \ (mod\ 7)$ for even m and $-1 \ (mod\ 7)$ for odd m.

So we have

$$
\begin{aligned}
a &= b_0 + b_1 \cdot 10^3 + b_2 \cdot 10^6 + \cdots + b_n \cdot 10^3 n \\
&= b_0 - b_1 + b_2 + \cdots + (-1)^n \cdot b_n \ (mod\ 7) \\
&= c \ (mod\ 7)
\end{aligned}
$$

Thus $a = c\ (mod\ 7)$. Thus if c is divisible by 7, so is a (since a and c have the same remainder when divided by 7).

The proof of why this method works for checking divisibility by 13 is the same, by replacing the 7's in the proof by 13's.

Finally, to show that $|c| \leq 999(n+1)$, notice that $|a+b| \leq |a| + |b|$ for any numbers a, b (by the Triangle Inequality). So $|a-b| = |a+(-b)| \leq |a| + |-b| = |a| + |b|$ for any positive integers a, b. So by applying the Triangle inequality repeatedly, we have

$$|c| = |b_0 - b_1 + b_2 + \cdots + (-1)_n \cdot b_n| \leq |b_0| + |b_1| + |b_2| + \cdot + |b_n|.$$

Since $b_m \leq 999$ for each m and we have $n+1$ terms, $|c| \leq 999(n+1)$. (In fact, there is a smaller upper bound for $|c|$, $999 \cdot \lceil \frac{n+1}{2} \rceil$.)

(As an aside, the trick to find out if a number is divisble by 37 is similar to the above method – we define c as the sum of b_m's instead of the above definition. If $c > 999$, repeat the process. Otherwise, multiply the leading digit of c by 111 and subtract the product from c. If the difference is one of -74, -37, 0, 37, 74, then the original number is divisible by 37.) $\square$