The 7 Degrees of 6,174

We begin this section by showing a little game involving 4-digit integers.

We start by picking any integer between 1 and 9999, so that there is at least one digit is different than the others (so multiples of 1111 are not allowed).

If the chosen number has 3 digits or fewer, we add zeros in the front until the number has 4 digits (so, for example, if we pick the number 496, we express it as 0496).

Then we write the digits of the original number in non-increasing order and call it A. We then write the digits of the original number in non-decreasing order and call it B. (In this example, $A = 9640$, $B = 469$, and the difference is 9171.)

We now subtract B from A and, if the difference is not 6174, repeat the steps mentioned in the last two paragraphs. After a few steps, we will always get 6174 as the difference. In fact, we reach the number 6174 after at most 7 iterations. (In the example, we reach 6174 after 3 iterations).

Proof: First, we choose a number between 1 and 9999 so that not all its digits are the same, and find A, B as defined above. So $A = 1000a + 100b + 10c + d$ and $B = 1000d + 100c + 10b + a$, where $9 \geq a \geq b \geq c \geq d \geq 0$, and $a > d$.

Now $A > B$, since $9 \geq a \geq b \geq c \geq d \geq 0$ and $a > d$.

$A - B$ is always a multiple of 9. The proof is as follows:

$$\begin{aligned} A - B &= 1000a + 100b + 10c + d - (1000d + 100c + 10b + a) \\ &= 999a + 90b - 90c - 999d \\ &= 9(111a + 10b - 10c - 111d) \end{aligned}$$

Since a, b, c, d are all integers, $A - B$ is divsible by 9.

Now we study the digits of $A - B$. In the following, I refer the digit representing thousands as the first digit.

The last digit of $A - B$ is $d - a$. Since $a > d$, $d - a < 0$. So when we subtract B from A, we have to reduce the third digit of A, c, by 1 and add 10 to $a - d$ so that the last digit of $A - B$ is between 1 and 9.

Likewise, since $b \geq c$ and we have reduced c by 1 when we subtract the last digit, the third digit of $A - B$ becomes $(c - 1) - b$. But since $b \geq c$, $b > c - 1$. So we have to reduce the second digit of A, b, by 1 and add 10 to $(c - 1) - b$.

Now, depending on the values of b and c, we may or may not need to reduce the first digit of A by 1 in order to have all digits of $A - B$ being non-negative.

If $b > c$, then $b - 1 \geq c$. So when we subtract the second digit of A by the second digit of B, the second digit of $A - B$ is non-negative, so we don't need to reduce the first digit of A by 1 before subtraction. Otherwise (when $b = c$), we reduce the first digit of A, a, by 1 and add 10 to $(b - 1) - c$, which yields 9.

So the digits of $A - B$ have the following form:

We get (1) when $b > c$ and (2) when $b = c$.

By adding the digits of $A - B$, we see that the sum is either 18 (in (1)) or 27 (in (2)).

	1000	100	10	1	Value
	a	b	c	d	A
-	d	c	b	a	B
	$a-d$	$b-1-c$	$9+c-b$	$10+d-a$	(1)
	$a-1-d$	$10+b-1-c$	$10+c-b-1$	$10+d-a$	(2)

Table 1:

Now we consider the sum of the middle two digits.

In case (1), the sum of those two digits is 8. Since the digits are positive, the two digits can be any of the following pairs: (0,8), (1,7), (2,6), (3,5), (4,4), in either order. So we have 9 ordered pairs of digits.

The sum of digits of the other two digits is 10. Since the digits are single-digit numbers, they can be any of the following pairs: (1,9), (2,8), (3,7), (4,6), (5,5), in either order. So we have 9 ordered pairs of digits.

So, for case (1), there are 81 possible digit combinations. Also, notice that 6,174 is among the 81 numbers.

The following table shows the digit pairs rearranged so that the digits are in non-increasing order.

	(1,9)	(2,8)	(3,7)	(4,6)	(5,5)
(0,8)	9810	8820	8730	8640	8550
(1,7)	9711	8721	7731	7641	7551
(2,6)	9621	8622	7632	6642	6552
(3,5)	9531	8532	7533	6543	5553
(4,4)	9441	8442	7443	6444	5544

Table 2:

In case (2), the second and third digits are both 9, as $b = c$.

Since the sum of digits in (2) is 27. The sum of the first and last digits is 9. So those two digits can be any of the following pairs: (0,9), (1,8), (2,7), (3,6), (4,5), in either order.

So the possible differences obtained in case (2) are: 999, 1998, 2997, 3996, 4995, 5994, 6993, 7992, 8991, and 9990.

If we rearrange the digits of the above numbers so that the digits are in non-increasing order, we obtain the following 30 numbers.

9990	9810	8820	8730	8640	8550
9981	9711	8721	7731	7641	7551
9972	9621	8622	7632	6642	6552
9963	9531	8532	7533	6543	5553
9954	9441	8442	7443	6444	5544

Table 3:

Now, we can define A and B as above to get Table 4, where number of steps denotes the number of iterations of the method described at the beginning of the section needed to obtain 6,174.

Since, after the first subtraction and the arrangement of digits in non-increasing order afterwards, we always get one of the 30 numbers in Table 3. So we will always obtain the number 6,174 by using the method described at the beginning of the section at most 7 times. □

A	B	$A-B$	Digits of $A-B$ written in non-increasing order	No. of steps
8532	2358	6174		1
7641	1467	6174		1
9711	1179	8532	8532	2
9621	1269	8352	8532	2
8820	288	8532	8532	2
8730	378	8352	8532	2
7533	3357	4176	7641	2
6642	2466	4176	7641	2
9981	1899	8082	8820	3
9963	3699	6264	6642	3
9810	189	9621	9621	3
6543	3456	3087	8730	3
9990	999	8991	9981	4
8622	2268	6354	6543	4
7731	1377	6354	6543	4
7443	3447	3996	9963	4
6552	2556	3996	9963	4
6444	4446	1998	9981	4
5553	3555	1998	9981	4
5544	4455	1089	9810	4
9972	2799	7173	7731	5
9954	4599	5355	5553	5
8721	1278	7443	7443	5
7632	2367	5265	6552	5
9531	1359	8172	8721	6
9441	1449	7992	9972	6
8640	468	8172	8721	6
8550	558	7992	9972	6
8442	2448	5994	9954	6
7551	1557	5994	9954	6

Table 4: